Homotopy choice in homotopy optimization framework

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Introduction

One of the approaches to optimization problems is homotopy framework. It helps to give local optimizer initial point that is close to local minimum of intermediate functions. Which should reduce amount of iterations and sometimes increase the probability of finding global minimum. Some initial experinemts shown that homotopy is also can ve useful in non-convex problems. The goal of this study is to research how the choice of homotopy affect optimization process and how it can deal with some famous non-convex problems.

Homotopy

Let target function be $f : \mathbb{R}^d \to \mathbb{R}$, then homotopy $h : \mathbb{R}^d \times [0, 1] \to \mathbb{R}$, and $h(\,\cdot\,, 1) = f(\,\cdot\,), h(\,\cdot\,, 0) = g(\,\cdot\,)$, where $g(\,\cdot\,)$ is a function that is easy to optimize. Futher $g(x) = x^* x$.

Restrictions

In case the target function can be any continious functional, the homotopy choice should be rectricted. In this study only $h(\,\cdot\,) = a(t)f(\,\cdot\,) + b(t)g(\,\cdot\,)$, where $a, b : [0, 1] \to [0, 1]$, $a(0) = 0, a(1) = 1, b(0) = 1, b(1) = 0$ are considered.

Experimental set

- Homotopies:
  - $h(t, \,\cdot\,) = tf(\,\cdot\,) + (1-t)g(\,\cdot\,)$ - linear
  - $h(t, \,\cdot\,) = t^2 f(\,\cdot\,) + (1-t)^2 g(\,\cdot\,)$ - square
  - $h(t, \,\cdot\,) = t^4 f(\,\cdot\,) + (1-t)^4 g(\,\cdot\,)$ - quad
  - $h(t, \,\cdot\,) = (0.5(2t^3 - 1) + 0.5)f(\,\cdot\,) + (1 - (0.5(2t^3 - 1) + 0.5))g(\,\cdot\,)$ - cubic
  - $h(t, x) = \mathbb{E}_{\delta x \in \mathbb{B}_{(1-t)}(0)}(x + \delta x)$ - smooth
- Target functions:
  - Levi's: $f(x, y) = \sin^2 3\pi x + (x-1)^2 \left(1 + \sin^2 3\pi y\right) + (y-1)^2 \left(1 + \sin^2 2\pi y\right)$
  - Rosenbrock: $f(x) = \sum_{i=1}^{n-1} \left[100 \left(x_{i+1} - x_i^2\right)^2 + (1 - x_i)^2\right]$
  - Himmelblau's: $f(x, y) = (x^2 + y - 11)^2 + (x + y^2 - 7)^2$
  - Easom: $f(x, y) = -\cos(x)\cos(y)\exp\left(-\left((x-\pi)^2 + (y-\pi)^2\right)\right)$
  - Cross-in-tray function: $f(x, y) = -10^{-4}\left[\left|\sin x \sin y \exp\left(\left|100 - \frac{\sqrt{x^2+y^2}}{\pi}\right|\right)\right| + 1\right]^{0.1}$
- Local minimizer: Powell

Algorithm

We solve this problem using HOM:

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Input: h, f, g, MAX_ITER, init

for iter = 1:MAX_ITER
init = Powell (h(f, g, t), init)
end
Output: min
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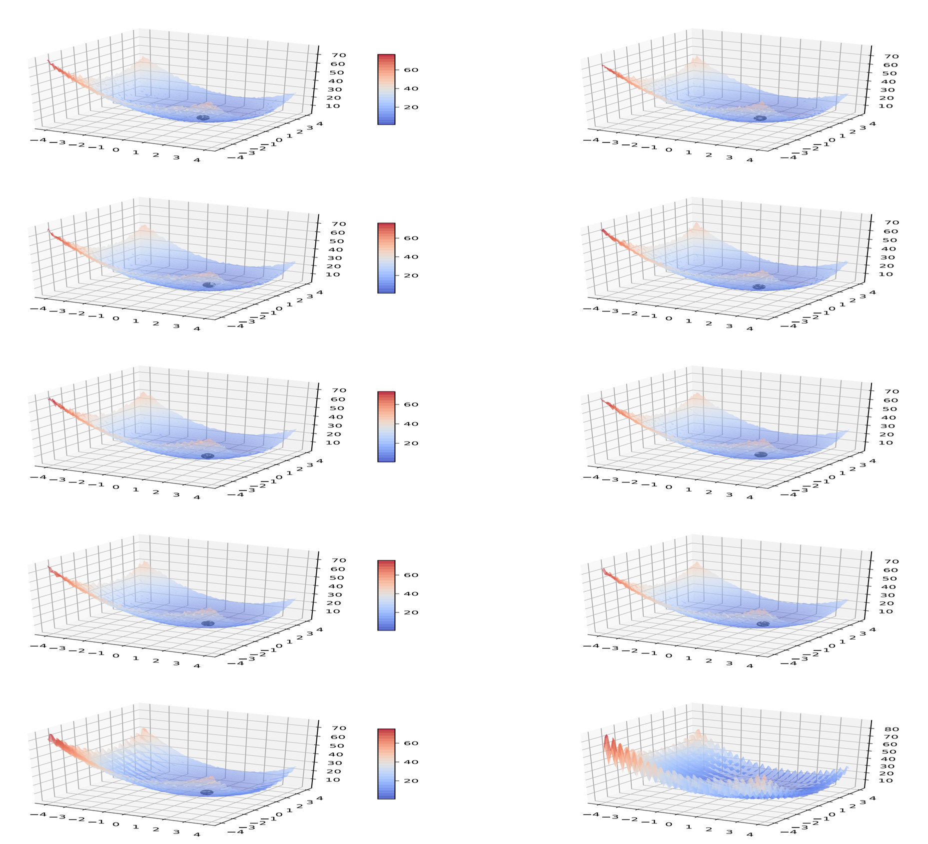
Example

$f = Levi's,\ h = smooth,\ MAXITER = 20$



Trajectory: $(0, 0), (0.99, 1.07), (0.99, 1.14), (1.05, 0.98), (0.98, 1.03), (0.96, 1.09), (0.96, 1.05), (1.02, 0.99), (1.15, 0.98), (1.04, 0.93), (0.92, 0.96), (0.95, 1.01), (1.00, 1.01), (0.97, 0.99), (0.88, 1.00), (0.87, 0.99), (0.86, 1.03), (0.96, 1.01), (0.98, 0.99), (0.99, 1.04), (1.0, 1.0))$

Convergence

Example shows that homotopy can be effective in some non-convex problems. It can be helpfull to reduce the amount of "non-convexity". But further sight into this idea requires new essenses in convex analysis. Therefore this study provides only experiments.

Experiment

In this research every HOM launch made 20 iterations. Metrics are the following: total amount of local minimizer iterations and proximity of each intermediate local minimum to target minimum.

Results

Total number of iterations local minimizer spent during HOM.

	linear	square	quad	cubic	smooth	no homo
Levi's	80	78	74	80	76	80
Rosenbrock	149	140	165	125	86	5
Himmelblau's	84	84	80	83	77	9
Easom	59	58	54	53	75	5
Cross	80	78	70	80	71	80

Another metric is mean square distance from initial guess to real global minimum. Which is more accurately evaluate the quality of homotopy, because it shows how well homotopy bring algorithm closer to global minimum.

	linear	square	quad	cubic	smooth
Levi's	4.45	4.52	4.6	3.6	0.54
Rosenbrock	4.56	4.57	4.57	4.56	5.66
Himmelblau's	4.27	7.05	9.25	3.7	0.44
Easom	19.4	19.4	19.4	19.4	13.6
Cross	7.85	7.05	6.67	7.89	1.03

Red items didn't converged to global minimum.

Graphs on github.com/higheroplane/HomotopyOpt

Conclusion

In this study some homotopies were compared on several "bad" functiouns by two metrics: total amount of iterations local minimizer did during HOM algorithm, and mean square of the distanse between initial guess and exact global minimum. This study also shiows that non-convex problems might be resolved by smothening and "convexening" target functions using homotopy framework.

References

[1] Nicholas Sun. Why convex homotopy is very useful in optimization: A possible theoretical explanation. *Journa of Uncertain Systems*, 2014.

[2] DIANNE P. O LEARY DANIEL M. DUNLAVY. Homotopy optimization methods for global optimization. *CS UMD*, 2005.