Minimax Interpolation

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Introduction

In this paper, the idea of approximation of Gaussian stationary processes and differentiable functions will be described. In particular, a theory of approximation of Gaussian processes based on the minimax covariance function will be derived. The quality of the approximators based on various covariance functions will be compared, and it will be shown that the processes based on the minimax covariance function show the best results.

General theory

Let $f(x)$ be a stationary Gaussian process on $\mathbb{R}^d$ with a covariance function

$$c(x) = \mathbb{E}(f(x_0+x) - \mathbb{E}f(x_0+x))(f(x_0) - \mathbb{E}f(x_0)).$$

Then the spectral density $F(\omega)$ is defined as

$$F(\omega) = \int_{\mathbb{R}^d} e^{2\pi i \omega^T x} c(x) dx$$

Suppose we know the values of the implementation of $f(cdot)$ on an infinite rectangular grid

$$D_H = \{x_k : x_k = H \cdot k, k \in \mathbb{Z}^d\}$$

where H is a diagonal matrix.
Interpolation error for area $\Omega_H = [0, h_1] \times \dots \times [0, h_d]$ is defined as:

$$\sigma_H^2(\tilde{f}, F) = \frac{1}{\mu(\Omega_H)} \int_{\mathbb{R}^d} \mathbb{E}[\tilde{f}(x) - f(x)]^2 dx$$

where $\mu(\Omega_H) = M_{i=1}^d h_i$ – Lebesgue measure Ω_H, $\tilde{f}(x)$ – interpolation $f(x)$.
We will consider $\tilde{f}(x)$ kind of

$$\tilde{f}(x) = \mu(\Omega_H) \sum_{x' \in D_H} K(x - x') f(x_k)$$

where $K(\cdot)$ – symmetric kernel.

Interpolation process

Consider the one-dimensional case when

$$\hat{K}(\omega) = \begin{cases} 1 - |w| \cdot h, & if \ \ |w| \le \frac{1}{h} \\ 0, & else \end{cases}$$

The recovered $\tilde{f}(x)$ process that minimizes the interpolation error is

$$\tilde{f}(x) = h \cdot \sum_{x_k \in D_H} K(x - x_k) f(x_k)$$

Finding the Fourier transform of the kernel $\hat{K}(\omega)$, we get the interpolation process $\tilde{f}(x)$:

$$\tilde{f}(x) = h^2 \cdot \sum_{x_k \in D_H} \frac{\sin^2(\frac{(x-x_k)\pi}{h})}{((x-x_k)\pi)^2} f(x_k)$$

$$x_k = h \cdot k, \quad k = 0, \pm 1, \pm 2, \dots$$

Minimax error

Define the set $\mathbb{F}(L, \lambda)$ of spectral densities $F(\omega)$ for a given $\lambda \in \mathbb{R}^d$ and $L > 0$:

$$\mathbb{F}(L, \lambda) = \{F : \mathbb{E} \sum_{i=1}^d \lambda_i^2 (\frac{\partial f_F(x)}{\partial x_i})^2 \le L, x \in \mathbb{R}^d\}$$

where $f(x) = f_F(x)$ is a Gaussian process with spectral density $F(omega)$, observed at the point $x \in \mathbb{R}^d$.

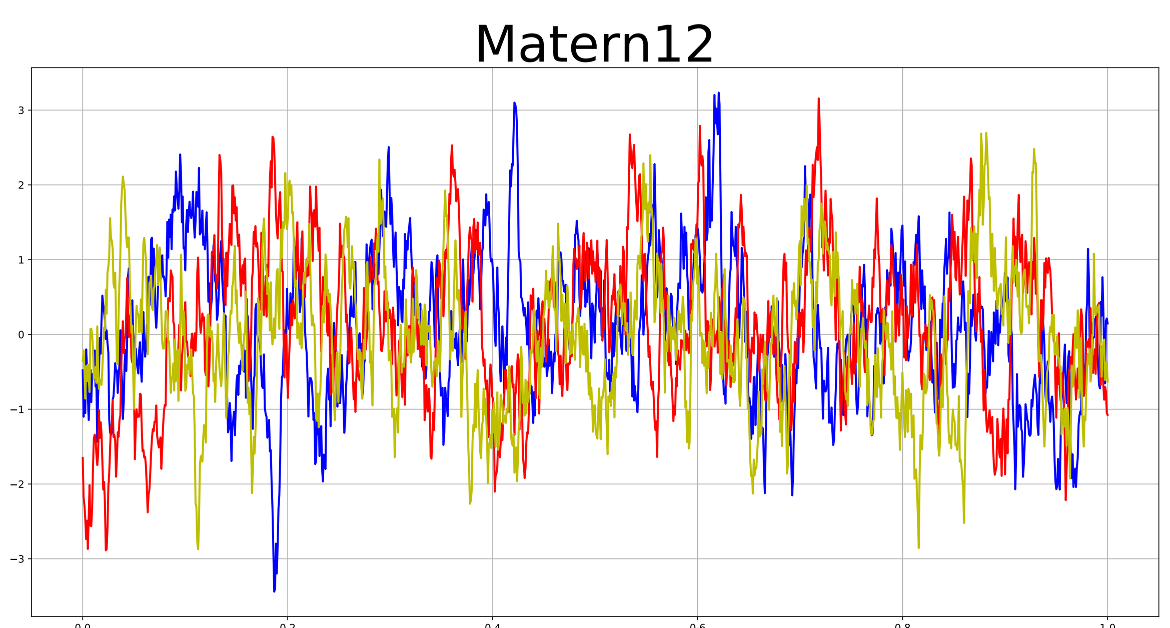
Determine the minimax interpolation error:

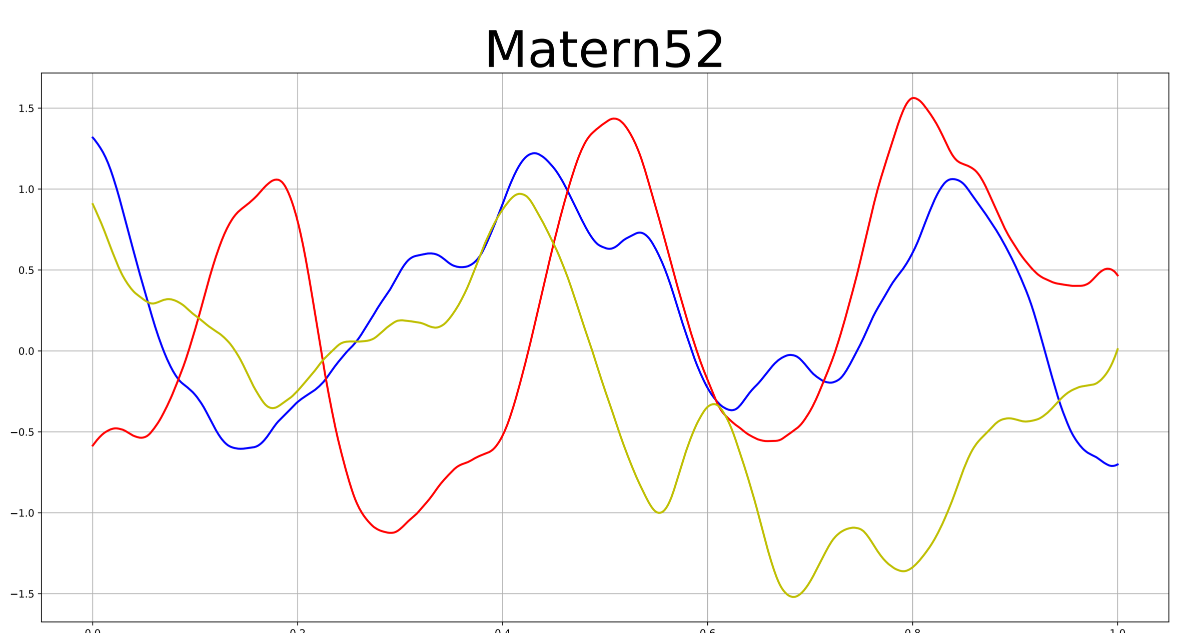
$$R^H(L, \lambda) = \inf_{\tilde{f}} \sup_{F \in \mathbb{F}(L,\lambda)} \sigma_H^2(\tilde{f}, F)$$

Process generation

The experiments examined the implementation of Gaussian processes generated by the following covariance functions. The same covariance functions are used to build regression models.

- $Matern_{1/2} : c(r) = \sigma^2 \exp(\frac{-r}{\rho})$,
- $Matern_{3/2} : c(r) = \sigma^2 (1 + \frac{\sqrt{3}r}{\rho}) \exp(\frac{-\sqrt{3}r}{\rho})$,
- $Matern_{5/2} : c(r) = \sigma^2 (1 + \frac{\sqrt{5}r}{\rho} + \frac{5r^2}{3\rho^2}) \exp(\frac{-\sqrt{5}r}{\rho})$.



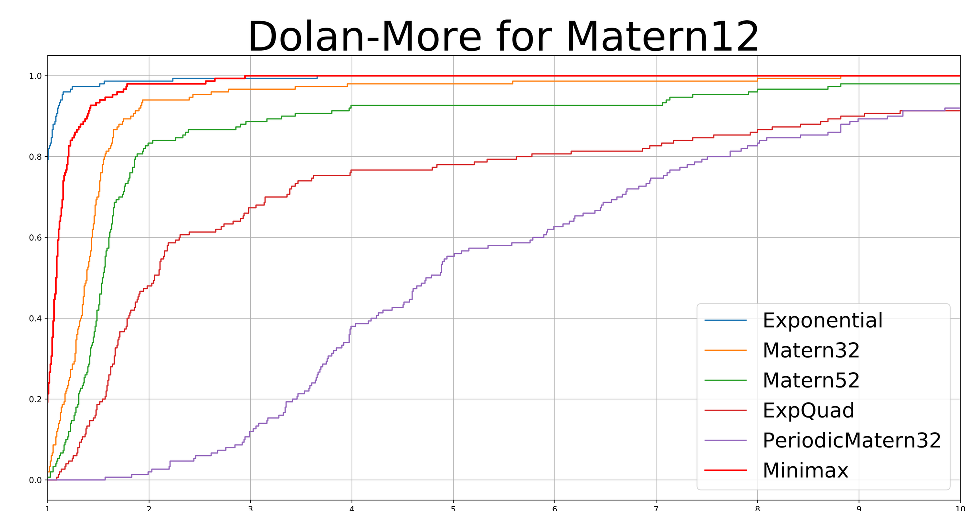


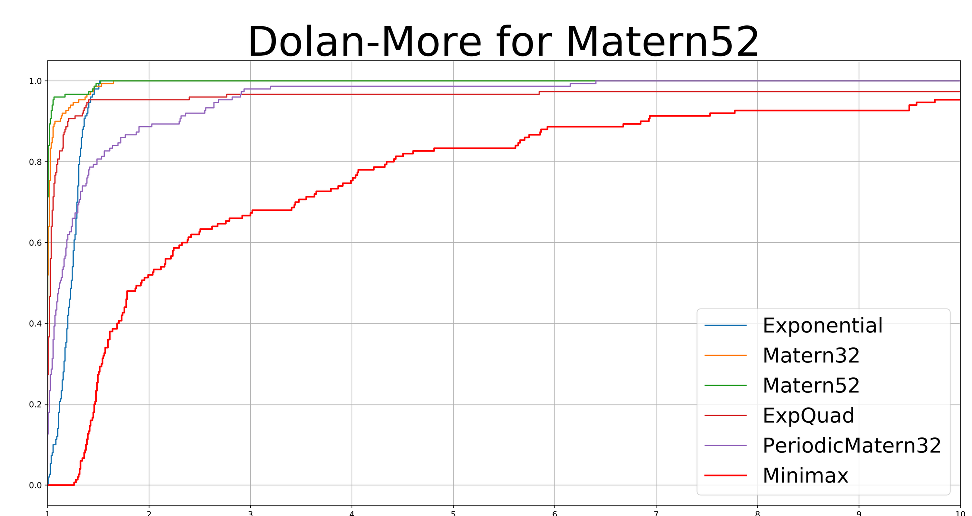
Computational experiments

Our task is to compare the error for the minimax covariance function and for other covariance functions commonly used to create regression models based on Gaussian processes.

Results

The figures show that with increasing smoothness of tasks, the quality of the minimax interpolation decreases as compared to other regression models. For the Matern12 covariance function, our model shows adequate results compared to other models, which was expected according to the theory. For a quadratic exponential function, on the contrary, the smoothness of the minimax covariance function too low results in a poor quality of the models obtained.





Conclusion

1. An interpolation Gaussian process type was obtained.
2. Our interpolation model performed better than other models on nonsmooth problems.
3. Also, our model has shown itself to be more resistant to noise.

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2. **Methods for constructing regression models of heterogeneous sources of data for industrial engineering**. Alexey Zaytsev